ON THE UNIVERSAL PROPERTY OF PIMSNER-TOEPLITZ C^*-ALGEBRAS AND THEIR CONTINUOUS ANALOGUES

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ABSTRACT. We consider C^*-algebras generated by a single Hilbert bi-module (Pimsner-Toeplitz algebras) and by a product systems of Hilbert bimodules. We give a new proof of a theorem of Pimsner, which states that any representation of the generating bimodule gives rise to a repre-sentation of the Pimsner-Toeplitz algebra. Our proof does not make use of the conditional expectation onto the subalgebra invariant under the dual action of the circle group. We then prove the analogous statement for the case of product systems, generalizing a theorem of Arveson from the case of product systems of Hilbert spaces.

1. INTRODUCTION

Let E be a Hilbert module over $\mathcal{A}$, equipped with a left action of $\mathcal{A}$ via adjointable operators. We shall refer to such an E as a Hilbert bi-module. We assume that E is full, i.e. $\overline{\langle E, E \rangle} = \mathcal{A}$. We make no further assumptions on the left action of $\mathcal{A}$. Let $\mathcal{B}$ be a C^*-algebra. A *covariant homomorphism* ψ of E into $\mathcal{B}$ is a $\mathbb{C}$-linear map $\psi_E : E \to \mathcal{B}$ along with a homomorphism $\psi_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ such that for all $e, f \in E$, $a, b \in \mathcal{A}$ we have $\psi_E(aeb) = \psi_{\mathcal{A}}(a)\psi_E(e)\psi_{\mathcal{A}}(b)$, $\psi_E(e)^*\psi_E(f) = \psi_{\mathcal{A}}(\langle e, f \rangle)$. In the sequel, we will write ψ for both $\psi_{\mathcal{A}}$ and ψ_E, when it causes no confusion. When $\mathcal{B} = \mathcal{B}(H)$ for a Hilbert space H, we call a covariant homomorphism a *representation*.

Remark 1.1. In [MS] what we call a *representation* is called an *isometric covariant* representation. Since we won't deal with other kinds of represen-tations considered in [MS], we shall use the shorter terminology.

We recall Pimsner's construction. Let $\mathcal{E} = \bigoplus_{n=0}^{\infty} E^{\otimes n}$ (where we take $E^{\otimes 0} = \mathcal{A}$). For $e \in E$, let $T_e \in \mathcal{B}(\mathcal{E})$ be given by $T_e(\xi) = e \otimes \xi$. The map sending

$$e \mapsto T_e, \ \mathcal{A} \ni a \mapsto \ \text{left multiplication by } a$$

is a covariant homomorphism of E into $\mathcal{B}(\mathcal{E})$. We let $\mathcal{T}_E$ be the C^*-subalgebra of $\mathcal{B}(\mathcal{E})$ generated by $\{T_e \mid e \in E\}$. We will lighten notation by identifying $e \in E$ with T_e, and a with multiplication by a in $\mathcal{T}_E$, when it does not lead to confusion.

Note that if $\pi : \mathcal{T}_E \to \mathcal{B}(H)$ is a representation, then the restrictions of π to E and $\mathcal{A}$ form a representation of E. Our goal in the first section will be

is to give a new proof of the following theorem – a restatement of a theorem of Pimsner ([P], Theorem 3.4) – which shows that any representation arises in this manner.

Theorem 1.2 (Pimsner)**.** *Let ψ be a representation of E on a Hilbert space H. The map $e \to \psi(e)$ extends to a homomorphism $\mathcal{T}_E \to \mathcal{B}(H)$.*

Remark 1.3. Pimsner's proof relies on the conditional expectation map of the algebra $\mathcal{T}_E$ onto the fixed point subalgebra for the dual action of $\mathbb{T}$, generalizing the proof for the Cuntz algebras from [Cu]. The motivation leading to the proof presented herein was to obtain the continuous analogue, Theorem 1.8 below. The continuous analogues, described below, admit an analogous action of $\mathbb{R}$, rather than $\mathbb{T}$. Thus, one cannot obtain a bounded expectation map by averaging the group action. We note that an unbounded expectation map has been used by Zacharias to study Arveson's spectral C^*-algebras in [Z]. We refer the reader to [A2] for more details on the spectral C^*-algebras, and to [HZ] for a recent survey.

In [H], we considered a certain continuous analogue of the algebras $\mathcal{T}_E$, generalizing to the context of Hilbert modules Arveson's spectral C^*-algebras (see [A2]). We recall the definitions.

Definition 1.4. Let $\mathcal{A}$ be a separable C^*-algebra. A *measurable bundle of Hilbert $\mathcal{A}$-bimodules over Ω, E*, is a collection $\{E_x \mid x \in \Omega\}$ of right Hilbert $\mathcal{A}$-modules with left actions via adjointable operators, along with a distinguished vector subspace Γ of $\Pi_{x \in \Omega} E_x$ (called the set of *measurable sections*) such that

(1) For any $\xi \in \Gamma$, $a \in \mathcal{A}$, the functions $x \mapsto \langle \xi(x), \xi(x) \rangle$, $x \mapsto \langle a\xi(x), \xi(x) \rangle$ are measurable (as functions $\Omega \mapsto \mathcal{A}$).
(2) If $\eta \in \Pi_{x \in \Omega} E_x$ satisfies that $x \mapsto \langle \xi(x), \eta(x) \rangle$ is measurable for all $\xi \in \Gamma$ then $\eta \in \Gamma$.
(3) There exists a countable subset $\xi_1, \xi_2, ...$ of Γ such that for all $x \in \Omega$, $\xi_1(x), \xi_2(x), ...$ are dense in E_x.

We refer the reader to the appendix of [H] for more details.

Definition 1.5. Let $\mathcal{A}$ be a separable C^*-algebra. A *product system of $\mathcal{A}$-bimodules E* is a measurable bundle of $\mathcal{A}$-bimodules over $\mathbb{R}_+$, along with a multiplication map $E \times E \to E$, which descends to an isomorphism $E_x \otimes_{\mathcal{A}} E_y \to E_{x+y}$ for all $x, y \in \mathbb{R}_+$ (where E_x is the fiber over x), and is measurable in the sense that if ξ is a measurable section and $e \in E_y$ then the sections $x \mapsto e\xi(x - y)$, $x \mapsto \xi(x - y)e$ (0 if $x < y$) are also measurable.

The elements $e \in E$ act on $\int_{\mathbb{R}_+}^{\oplus} E_x dx$ on the left as adjointable operators, which we denote W_e, or by abuse of notation, just e. Note that $\|e\|_{E_x} \geq \|W_e\|_{\mathcal{B}(\int_{\mathbb{R}_+}^{\oplus} E_x dx)}$. Denote by $L^1(E)$ the space of measurable sections ξ that satisfy $\int_{\mathbb{R}_+} \|\xi(x)\| dx < \infty$.

Definition 1.6. For $f \in L^1(E)$ we define $W_f \in \mathcal{B}\left(\int_{\mathbb{R}_+}^{\oplus} E_x dx\right)$ by

$$W_f = \int_{\mathbb{R}_+} W_{f(x)} dx$$

We denote by $\mathcal{W}_E$ the C^*-subalgebra of $\mathcal{B}\left(\int_{\mathbb{R}_+}^{\oplus} E_x dx\right)$ generated by

$$\{W_f \mid f \in L^1(E)\}$$

We refer the reader to [H] for examples, and a discussion of the K-theory of $\mathcal{W}_E$.

Definition 1.7. Let E be a product system of $\mathcal{A}$-bimodules. A *representation* ψ of E on H is a map $\psi_E : E \to \mathcal{B}(H)$, along with a representation $\psi_\mathcal{A} : \mathcal{A} \to \mathcal{B}(H)$ such that
  (1) The restriction of ψ to each fiber of E is a representation of the fiber.
  (2) For any $e, f \in E$, $\psi(e)\psi(f) = \psi(ef)$.
  (3) If ξ is a measurable section of E then $x \mapsto \psi(\xi(x))$ is a weakly measurable function.
  (4) $\overline{\bigcup_{x>0} \psi(E_x)H} = \overline{\psi(A)H}$.

If $x \mapsto f(x)$ is a measurable section of E satisfying $\int_{\mathbb{R}_+} \|f(x)\| dx < \infty$ and ψ is a representation of E on H, then we have an integrated form of the representation

$$\psi(f) = \int_{\mathbb{R}_+} \psi(f(x)) dx$$

Our goal in the second part of this paper will be to prove the following continuous analogue of Theorem 1.2.

Theorem 1.8. *Let ψ be a representation of E on a Hilbert space H. The map $W_f \to \psi(f)$ extends to a homomorphism $\mathcal{W}_E \to \mathcal{B}(H)$.*

This theorem generalizes a theorem of Arveson ([A2], Theorem 4.6.6) from the case of product systems of Hilbert spaces. Specializing our proof below to the case of Hilbert spaces will give a simpler approach to Arveson's theorem.

2. The discrete case – proof of Theorem 1.2

Definition 2.1. Let ψ, ρ be two representations of E on H. We say that ψ *majorizes* ρ, and write $\psi \succ \rho$, if for any $e_1, ..., e_n \in E$ and any polynomial p in $2n$ non-commuting variables, we have

$$\|p(\psi(e_1), ..., \psi(e_n), \psi(e_1)^*, ..., \psi(e_n)^*)\| \geq \|p(\rho(e_1), ..., \rho(e_n), \rho(e_1)^*, ..., \rho(e_n)^*)\|$$

In other words, $\psi \succ \rho$ if there is a (necessarily unique) homomorphism $C^*(\{\psi(e) \mid e \in E\}) \to C^*(\{\rho(e) \mid e \in E\})$ which satisfies $\psi(e) \mapsto \rho(e)$ for all $e \in E$.

If $\psi \succ \rho$ and $\psi \prec \rho$, we write $\psi \approx \rho$.

We say that $T \succ \psi$ if the map $T_e \to \psi(e)$ extends to a homomorphism $\mathcal{T}_E \to C^*(\{\psi(e) \mid e \in E\})$, i.e. if

$$\|p(\psi(e_1), ..., \psi(e_n), \psi(e_1)^*, ..., \psi(e_n)^*)\| \leq \|p(e_1, ..., e_n, e_1^*, ..., e_n^*)\|_{\mathcal{B}(\mathcal{E})}$$

Thus Theorem 1.2 states that $T \succ \psi$ for any representation ψ of E.

Note that the relation $\succ$ is clearly transitive.

We first recall the following lemma (noted in [MS] and in references therein). The proof is straightforward.

Lemma 2.2. *Let ψ be a representation of E on H. Regarding H as a right $\mathcal{A}$-module via ψ, we form the tensor product $E \otimes_{\mathcal{A}} H$ to obtain a Hilbert space. The contraction map*

$$e \otimes \xi \mapsto \psi(e)\xi \quad e \in E, \, \xi \in H$$

is well defined and extends to an isometry

$$E \otimes_{\mathcal{A}} H \to H$$

If ψ is a representation of E, $n > 0$, then we can define a representation of $E^{\otimes n}$ by $e_1 \otimes \cdots \otimes e_n \mapsto \psi(e_1)\psi(e_2) \cdots \psi(e_n)$. We will denote this representation by ψ as well.

We may assume without loss of generality that $\psi_{\mathcal{A}}$ is non-degenerate, and we make this assumption throughout (i.e., we assume throughout that $\overline{\psi(\mathcal{A})H} = H$).

If π is a representation of $\mathcal{A}$ on a Hilbert space H, then we can define a representation $T \otimes_{\mathcal{A}} 1$ of E on $\mathcal{E} \otimes_{\pi} H$ (this is called an *induced* representation in [MS]). We note that clearly, $T \succ T \otimes_{\mathcal{A}} 1$. The following lemma generalizes the fact that any isometry S which satisfies $S^n S^{n*} \to 0$ in the strong operator topology is unitarily equivalent to a direct sum of copies of the unilateral shift on ℓ^2. The reader can find a proof in [MS].

Lemma 2.3. *Let ψ be a representation of E on H, such that $\bigcap_{n>0} \overline{\psi(E^{\otimes n})H} = \{0\}$. let $H_0 = (\psi(E)H)^{\perp}$.*

(1) *H_0 is invariant for $\psi(\mathcal{A})$.*

(2) *Let $H_n = \overline{\psi(E^{\otimes n})H_0}$. We have $H = \bigoplus_{n=0}^{\infty} H_n$.*

(3) *For any n, $W_n : E^{\otimes n} \otimes_{\mathcal{A}} H_0 \to H_n$ given by the contraction*

$$W_n(e_1 \otimes \cdots \otimes e_n \otimes \xi) = \psi(e_1)\psi(e_2) \cdots \psi(e_n)\xi$$

is a well defined unitary operator (where for W_0 is the contraction $a \otimes \xi \mapsto \psi(a)\xi$, $a \in E^{\otimes 0} = \mathcal{A}$).

(4) *$W = \bigoplus_{n=0}^{\infty} W_n : \mathcal{E} \otimes_{\mathcal{A}} H_0 \to H$ is a unitary operator which satisfies*

$$W(T_e \otimes_{\mathcal{A}} 1) = \psi(e)W$$

for all $e \in E$, i.e. it implements a unitary equivalence between the covariant representations $T \otimes_{\mathcal{A}} 1_{H_0}$ and ψ.

Corollary 2.4. *Let ψ be as in Lemma 2.3, then $T \succ \psi$.*

Now let ψ be any (non-degenerate) representation of E on H. By Corollary 2.4, to prove Theorem 1.2, it suffices to show that ψ is majorized by a representation which satisfies the condition of Lemma 2.3.

For any $\lambda \in \mathbb{T}$, we define a representation ψ_λ, given by $\psi_\lambda(e) = \lambda\psi(e)$, $\psi_\lambda(a) = \psi(a)$, $e \in E$, $a \in \mathcal{A}$. We can now form a direct integral to obtain a representation $\bar{\psi}$ on $H \otimes L^2(\mathbb{T})$, given by

$$\bar{\psi} = \int_\mathbb{T}^\oplus \psi_\lambda d\lambda$$

Since $\psi_\lambda(e) \to \psi(e)$ as $\lambda \to 1$ for all e (in norm), we can easily see that $\bar{\psi} \succ \psi$.

Let U be the bilateral shift on $\ell^2(\mathbb{Z})$. We form a representation $\tilde{\psi} = \psi \otimes U$ of E on $H \otimes \ell^2(\mathbb{Z})$ by $\tilde{\psi}(e) = \psi(e) \otimes U$, $\tilde{\psi}(a) = a \otimes 1$, $e \in E$, $a \in \mathcal{A}$. Applying the Fourier transform to the second variable shows that $\bar{\psi}$ and $\tilde{\psi}$ are unitarily equivalent. (i.e. they are intertwined by a unitary). In particular, we have $\bar{\psi} \approx \tilde{\psi}$.

Denote by P_+ the projection of $H \otimes \ell^2(\mathbb{Z})$ onto $H \otimes \ell^2(\mathbb{N})$. We denote by ψ_+ the restriction of $\tilde{\psi}$ to the invariant subspace $H \otimes \ell^2(\mathbb{N})$, i.e. $\psi_+(e) = \tilde{\psi}(e)P_+$ (where here we will think of ψ_+ as both a representation on $H \otimes \ell^2(\mathbb{N})$ and as a (degenerate) representation on $H \otimes \ell^2(\mathbb{Z})$). Denote by S the unilateral shift on $\ell^2(\mathbb{N})$, and let $V = 1_H \otimes S$.

Observation 2.5. *For any k, any polynomial $p(x_1, ..., x_{2k})$ in $2k$ non-commuting variables and any $e_1, ..., e_k \in E$, and any $m > \deg(p)$, we have*

$$V^{m*}p(\psi_+(e_1), ..., \psi_+(e_k)^*)V^m - P_+p(\tilde{\psi}(e_1), ..., \tilde{\psi}(e_k)^*)P_+ = 0$$

We leave the straightforward verification to the reader.

Lemma 2.6. *If A is in the $*$-algebra generated by*

$$\{\tilde{\psi}(e) \,|\, e \in E\}$$

then

$$\|P_+AP_+\| = \|A\|$$

Proof. Note that any operator of the form $\tilde{\psi}(e)$ commutes with all operators of the form $1_H \otimes U^n$, $n \in \mathbb{Z}$. Therefore A commutes with $1 \otimes U^n$, $n \in \mathbb{Z}$ as well. Let P_n denote the projection onto $H \otimes \ell^2(\{n, n+1, ...\})$ (so $P_0 = P_+$). We have $(1 \otimes U^m)P_n(1 \otimes U^m)^* = P_{n+m}$ for all $n, m \in \mathbb{Z}$, and therefore we have

$$(1 \otimes U^n)P_+AP_+(1 \otimes U^n)^* = P_nAP_n$$

so $\|P_+AP_+\| = \|P_nAP_n\|$ for all $n \in \mathbb{Z}$. Since $P_n \to 1_{\ell^2(\mathbb{Z})}$ as $n \to -\infty$ in the strong operator topology, we have

$$\|P_+AP_+\| = \lim_{n \to -\infty} \|P_nAP_n\| = \|A\|$$

as required. $\qquad\square$

Corollary 2.7. $\psi_+ \succ \tilde{\psi}$.

Proof. Let $e_1, ..., e_k \in E$, and let $p(x_1, ..., x_{2k})$ be a polynomial in $2k$ non-commuting variables. Since the V_m are isometries, we have

$$\|V^{m*}p(\psi_+(e_1), ..., \psi_+(e_k)^*)V^m\| \leq \|p(\psi_+(e_1), ..., \psi_+(e_k)^*)\|$$

so by Observation 2.5, we have

$$\left\|P_+p(\tilde{\psi}(e_1), ..., \tilde{\psi}(e_k)^*)P_+\right\| \leq \|p(\psi_+(e_1), ..., \psi_+(e_k)^*)\|$$

and by Lemma 2.6,

$$\left\|P_+p(\tilde{\psi}(e_1), ..., \tilde{\psi}(e_k)^*)P_+\right\| = \left\|p(\tilde{\psi}(e_1), ..., \tilde{\psi}(e_k)^*)\right\|$$

$\square$

Proof of Theorem 1.2. Note that ψ_+ satisfies the conditions of Lemma 2.3. It therefore suffices to show that $\psi_+ \succ \psi$, and indeed, we saw that $\psi_+ \succ \tilde{\psi}$ and $\tilde{\psi} \succ \psi$. $\square$

Remark 2.8. The proof in this section was obtained in the course of the author's dissertation work under the supervision of W.B. Arveson, and is motivated by ideas from [A2].

3. The continuous case – proof of Theorem 1.8

The approach here will differ from the proof above for the discrete case, in that we do not have a continuous analogue of Lemma 2.3 (see Remark 3.9 below). Aside for that, we shall follow a similar path.

We begin by giving the analogue of Definition 2.1.

Definition 3.1. Let ψ, ρ be two representations of a product system E (over $\mathcal{A}$) on H. We say that ψ *majorizes* ρ, and write $\psi \succ \rho$, if for any $f_1, ..., f_n \in L^1(E)$ and any polynomial p in $2n$ non-commuting variables, we have

$$\|p(\psi(f_1), ..., \psi(f_n), \psi(f_1)^*, ..., \psi(f_n)^*)\| \geq \|p(\rho(f_1), ..., \rho(f_n), \rho(f_1)^*, ..., \rho(f_n)^*)\|$$

In other words, $\psi \succ \rho$ if there is a (necessarily unique) homomorphism $C^*(\{\psi(f) \mid f \in L^1(E)\}) \to C^*(\{\rho(f) \mid f \in L^1(E)\})$ which satisfies $\psi(f) \mapsto \rho(f)$ for all $f \in L^1(E)$.

If $\psi \succ \rho$ and $\psi \prec \rho$, we write $\psi \approx \rho$.

We say that $W \succ \psi$ if the map $W_f \to \psi(f)$ extends to a homomorphism $\mathcal{W}_E \to C^*(\{\psi(f) \mid f \in L^1(E)\})$.

As in the discrete case, the relation $\succ$ is clearly transitive. Theorem 1.8 states that $W \succ \psi$ for any representation ψ of E.

As in the discrete case, we may assume without loss of generality that $\psi_{\mathcal{A}}$ is non-degenerate, and we make this assumption throughout.

Definition 3.2. Let ψ be a representation of E on H. A subspace H' of H is said to be *invariant* for ψ if $\psi(E_x)H' \subseteq H'$ (for all $x > 0$) and $\psi(\mathcal{A})H' \subseteq H'$. H' will be said to be *reducing* if it is invariant, and furthermore $\psi(e)^*H' \subseteq H'$ for all $e \in E$.

Let ψ be a representation of E on H, and let H' be invariant for ψ, then we have a representation of E on H' by restriction. Let P be the projection onto H', and let ψ' denote the restriction, then $\psi'(f) = \psi(f)P$. Notice that if H' is furthermore reducing, then $\psi \succ \psi'$.

We will make use of the following approximation lemma. The proof is straightforward, and we leave it to the reader.

Lemma 3.3. *Let ψ be a representation of E on H. Suppose that there is a sequence of projections $P_n \to 1$ in the strong operator toplogy, such that $P_n H$ is invariant for ψ for all n. Denote by ψ_n the restricted representation of ψ to $P_n H$. For any polynomial $p(x_1, ..., x_{2k})$ in $2k$ non-commuting variables and $f_1, ..., f_k \in L^1(E)$, if $\|p(\psi_n(f_1), ..., \psi_n(f_k), \psi_n(f_1)^*, ..., \psi_n(f_k)^*)\| \leq M$ for all n (for some constant M), then $\|p(\psi(f_1), ..., \psi(f_k), \psi(f_1)^*, ..., \psi(f_k)^*)\| \leq M$.*

Consequently, if ρ is a representation of E such that $\rho \succ \psi_n$ for all n then $\rho \succ \psi$.

If π is a representation of $\mathcal{A}$ on H, we can form a representation $W \otimes_{\mathcal{A}} 1$ of $\mathcal{W}_E$ on $\int_{\mathbb{R}_+}^{\oplus} E_x dx \otimes_\pi H$, as in the discrete case. We clearly have $W \succ W \otimes_{\mathcal{A}} 1$.

Let $S_x : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ denote the unilateral shift semigroup. We form a representation ψ_+ on $H \otimes L^2(\mathbb{R}_+)$ by $\psi_+(e) = \psi(e) \otimes S_x$ ($e \in E_x$), $\psi_+(a) = \psi(a) \otimes 1$. The following Lemma is a straightforward generalization, with a small improvement, of an argument in the proof of [A2], Theorem 4.4.3. We include a full proof for the reader's convenience.

Lemma 3.4.

(1) *Let ψ be a (non-degenerate) representation of E on a Hilbert space H. There is a unique isometry*
$$C : \int_{\mathbb{R}_+}^{\oplus} E_x dx \otimes_{\mathcal{A}} H \to H \otimes L^2(\mathbb{R}_+) \cong L^2(\mathbb{R}_+, H)$$
satisfying
$$C(f \otimes \xi)(x) = (\psi(f(x))\xi)$$
for any $\xi \in H$, $f \in L^1(E)$ such that $\int_{\mathbb{R}_+} \|f(x)\|^2 dx < \infty$ (those are dense both in $L^1(E)$ and in $\int_{\mathbb{R}_+}^{\oplus} E_x dx$).

The range of C is $H_\# = \{\xi \in L^2(\mathbb{R}_+, H) \mid \xi(x) \in \overline{\psi(E_x)H} \text{ a.e. } x\}$.

(2) *$H_\#$ is invariant for the representation ψ_+. Denote the restriction of ψ_+ to $H_\#$ by $\psi_\#$. C implements a unitary equivalence between $\psi_\#$ and $W \otimes_{\mathcal{A}} 1$ of E on $\int_{\mathbb{R}_+}^{\oplus} E_x dx \otimes_{\mathcal{A}} H$.*

Proof. C is well defined on the given domain, which is total in $\int_{\mathbb{R}_+}^{\oplus} E_x dx \otimes_{\mathcal{A}} H$. To show that C extends to an isometry, it suffices to check inner products on those vectors. So, for $f, g \in L^1(E)$ such that $\int_{\mathbb{R}_+} \|f(x)\|^2 dx < \infty, \int_{\mathbb{R}_+} \|g(x)\|^2 dx < \infty$, and $\xi, \eta \in H$, we have:

$\langle \psi(f(x))\xi, \psi(g(x))\eta \rangle_H = \langle \xi, \psi(f(x))^* \psi(g(x))\eta \rangle_H = \langle \xi, \psi(\langle f(x), g(x) \rangle_{\mathcal{A}})\eta \rangle_H =$

$$= \langle \psi(f(x)) \otimes_{\mathcal{A}} \xi, \psi(g(x)) \otimes_{\mathcal{A}} \eta \rangle_{E_x \otimes_{\mathcal{A}} H}$$

and now integrating both sides dx gives the required identity.

To prove that C has the required range, we first note that the range of C is clearly contained in $H_\#$. For the converse, suppose $g \in H_\#$ is orthogonal to the range of C. We must prove that $g = 0$.

Let $e_1(x), e_2(x), \dots$ be a sequence of measurable sections of E such that for all x, $\{e_1(x), e_2(x), \dots\}$ are total in E_x (we are guaranteed the existence of such sequence by the definition of a product system). We may assume that all those sections are bounded, without loss of generality. Let $u_1, u_2, \dots$ be a dense sequence in $L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. Let $\xi_1, \xi_2, \dots$ be a dense sequence in H. So, for all m, n, p, we have:

$$\int_0^\infty u_m(x) \langle \psi(e_n(x))\xi_p, g(x) \rangle = 0$$

Since $u_1, u_2, \dots$ are dense in $L^2(\mathbb{R}_+)$, we see that $\langle \psi(e_n(x))\xi_p, g(x) \rangle = 0$ a.e. x. Therefore, we have for all n, p and a.e. x, $\langle \psi(e_n(x))\xi_p, g(x) \rangle = 0$, and therefore, $g(x) = 0$ a.e., as required.

Finally, we need to check that $C(W_e \otimes_{\mathcal{A}} 1) = (\psi(e) \otimes S_x)C$ for all $e \in E_x$, $x \in \mathbb{R}_+$. It suffices to check this for vectors of the form $f \otimes_{\mathcal{A}} \xi$ as in the statement. Indeed, $C(W_e \otimes_{\mathcal{A}} 1)(f \otimes_{\mathcal{A}} \xi)(y) = C(ef \otimes_{\mathcal{A}} \xi)(y) = \psi((e \cdot f)(y))\xi = \psi(e \cdot f(y-x))\xi = \psi(e)\psi(f(y-x))\xi = (\psi(e) \otimes S_x)(C(f \otimes_{\mathcal{A}} \xi))$, as required (where $f(y-x)$ is understood to mean 0 if $x > y$). $\qquad\square$

Lemma 3.5. *Let ψ be a representation of E on H. Let ψ_+, $\psi_\#$, $H_\#$ be as in Lemma 3.4, then $\psi_+ \approx \psi_\#$.*

Proof. Since $H_\#$ is a reducing subspace, we have $\psi_+ \succ \psi_\#$. Thus it remains to show that $\psi_\# \succ \psi_+$. By Lemma 3.3, it suffices to exhibit projections $P_\varepsilon \in \mathcal{B}(L^2(\mathbb{R}_+, H))$ such that $P_\varepsilon(L^2(\mathbb{R}_+, H))$ is invariant for ψ_+, $P_\varepsilon \to 1$ as $\varepsilon \to 0$ (in the strong operator topology), and $\psi_\# \succ \psi_\varepsilon$ where ψ_ε denotes the restriction of ψ_+ to $P_\varepsilon(L^2(\mathbb{R}_+, H))$.

Denote $H_x = \overline{\psi(E_x)H}$. Let $K_\varepsilon = \{\xi \in L^2(\mathbb{R}_+, H) \mid \xi(x) \in H_x \ominus H_{x+\varepsilon}\} \subseteq H_\#$. K_ε is reducing for $\psi_\#$ (and for ψ_+). Denote the restriction of $\psi_\#$ to K_ε by $\psi_\#^\varepsilon$.

Now, for $n = 1, 2, \dots$, let $K_\varepsilon^n = \{\xi \in L^2(\mathbb{R}_+, H) \mid \xi(x) \in H_{x-n\varepsilon} \ominus H_{x-(n-1)\varepsilon}\}$. Note that the K_ε^n are mutually orthogonal, and are all orthogonal to $H_\#$.

K_ε^n is invariant (but not reducing) for ψ_+. Note that if $\xi \in K_\varepsilon^n$ then $\xi(x) = 0$ for a.e. $x \leq n\varepsilon$. Let ψ_ε^n denote the restriction of ψ_+ to K_ε^n.

Define $U_\varepsilon^n : K_\varepsilon \to K_\varepsilon^n$ by $U_\varepsilon^n(\xi)(x) = \xi(x - n\varepsilon)$ (where $\xi(x - n\varepsilon)$ is understood to be 0 if $x \leq n\varepsilon$). It is easy to check that U_ε^n is unitary, and implements a unitary equivalence between $\psi_\#^\varepsilon$ and ψ_ε^n.

Let $H_+^\varepsilon = H_\# \oplus \bigoplus_{n=1}^\infty K_\varepsilon^n \subseteq L^2(\mathbb{R}_+, H)$. Note that this space is invariant for ψ_+. Let P_ε be the projection onto the H_+^ε, and ψ_ε the restriction of ψ_+ to H_+^ε. So $P_\varepsilon \to 1$ (since, for example, the range of P_ε contains $L^2(\mathbb{R}_+, H_\varepsilon)$), and $\psi_\# \succ \psi_\varepsilon$ for all ε, which is what we needed. $\qquad\square$

We may now proceed as in the discrete case. For $\lambda \in \mathbb{R}$, we define a representation ψ_λ by $\psi_\lambda(e) = e^{ix\lambda}\psi(e)$, $\psi_\lambda(a) = \psi(a)$. We form a representation $\bar{\psi}$ on $H \otimes L^2(\mathbb{R})$, by

$$\bar{\psi} = \int_{\mathbb{R}}^{\oplus} \psi_\lambda d\lambda$$

and since $\psi_\lambda(e) \to \psi(e)$ as $\lambda \to 0$ for all e (in norm), we have $\bar{\psi} \succ \psi$.

Let U_x be the bilateral shift group on $L^2(\mathbb{R})$. We form a representation $\tilde{\psi}$ of E on $H \otimes L^2(\mathbb{R})$ by $\tilde{\psi}(e) = \psi(e) \otimes U_x$ $(e \in E_x)$, $\tilde{\psi}(a) = a \otimes 1$. Using the Fourier transform, we see that $\bar{\psi}$ and $\tilde{\psi}$ are unitarily equivalent, so $\tilde{\psi} \succ \psi$.

Let $V_x = 1_H \otimes S_x$, and let P_+ the projection of $H \otimes L^2(\mathbb{R})$ onto $H \otimes L^2(\mathbb{R}_+)$

The following are immediate analogues of Observation 2.5 and Lemma 2.6 above (and immediate generalizations of 4.5.3 and 4.5.4 in [A2]). We leave the simple proofs to the reader.

Observation 3.6. *For any k and any polynomial $p(x_1, ..., x_{2k})$ in $2k$ non-commuting variables and any $f_1, ..., f_k \in L^1(E)$, We have*

$$\lim_{x \to \infty} \left\| V_x^* p(\psi_+(f_1), ..., \psi_+(f_k)^*)V_x - P_+ p(\tilde{\psi}(f_1), ..., \tilde{\psi}(f_k)^*)P_+ \right\| = 0$$

Lemma 3.7. *If A is in the $*$-algebra generated by*

$$\{\tilde{\psi}(f) \,|\, f \in L^1(E)\}$$

then

$$\|P_+ A P_+\| = \|A\|$$

Corollary 3.8. $\psi_+ \succ \tilde{\psi}$.

Proof of Theorem 1.8. Let ψ be a (non-degenerate) representation of E on H. We want to show that $W \succ \psi$. So, $W \succ W \otimes_{\mathcal{A}} 1$, $W \otimes_{\mathcal{A}} 1 \approx \psi_\#$ (by Lemma 3.4), $\psi_\# \approx \psi_+$ (by Lemma 3.5), $\psi_+ \succ \tilde{\psi}$ (Corollary 3.8), and $\tilde{\psi} \succ \psi$ (as remarked above), concluding the argument. $\qquad\square$

Remark 3.9. There is a continuous analogue of the Wold decomposition, due to Cooper ([Co]), which states that if S_x $(x > 0)$ is a strongly continuous semigroup of isometries on a Hilbert space, then S_x is unitarily equivalent to a direct sum of a one-parameter unitary group and copies of the unilateral shift semigroup on $L^2(\mathbb{R}_+)$.

Unlike the case of a single bimodule, this does not quite generalize to product systems. There is an approximate version, due to Arveson ([A1, A2] for product systems of Hilbert spaces. Arveson's theorem states the following. Let E is a product system of Hilbert *spaces*, and let ψ be a representation of E on H such that $\bigcap_{x>0} \overline{\psi(E_x)H} = \{0\}$. For any $\varepsilon > 0$, let $H_\varepsilon = \overline{\psi(E_\varepsilon)H}$, and let ψ_ε be the restriction of ψ to H_ε, then ψ_ε is unitarily equivalent to a direct sum of the regular representation of E (on $\int_{\mathbb{R}_+}^{\oplus} E_x dx$, which is a Hilbert space here). However, Arveson showed in [A1] that H_ε cannot be replaced by H in the theorem. This theorem was used by Arveson to prove the special case of Theorem 1.8 for Hilbert spaces. We do not know

if the generalization of Arveson's theorem to the case of Hilbert modules holds.

References

[A1]   Arveson, W.B., Continuous analogues of Fock space. III. Singular states. *J. Operator Theory* **22** (1989), no. 1, 165–205.

[A2]   Arveson, W.B., *Noncommutative Dynamics and E-semigroups*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[Co]   Cooper, J. L. B., One-parameter semigroups of isometric operators in Hilbert space. *Ann. of Math. (2)* **48** (1947), 827–842.

[Cu]   Cuntz, J., Simple C^*-algebras generated by isometries. *Comm. Math. Phys.* **57** (1977), no. 2, 173–185.

[H]   Hirshberg, I., C^*-algebras of Hilbert module product systems, to appear, *J. Reine. Angew. Math.*

[HZ]   Hirshberg, I. and Zacharias, J., On the structure of spectral algebras and their generalizations, to appear, *Contemporary Mathematics*

[MS]   Muhly, P., and Solel, B., Tensor algebras, induced representations and the Wold decomposition, *Can. J. Math.* **51** (1999), no. 4, 850–880.

[P]   Pimsner, M.V., A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$. *Free probability theory (Waterloo, ON, 1995)*, 189–212, Fields Inst. Commun., **12**, Amer. Math. Soc., Providence, RI, 1997.

[Z]   Zacharias, J., Pure infiniteness of spectral algebras. *J. Funct. Anal.* **178** (2000), no. 2, 381–395.

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